inversely proportional to b_m ($D_{\alpha\beta} \propto 1/b_m$) in the region V_m, then Eq. (11) can be further simplified, such as

$$\frac{\partial u_i}{\partial b_m} = -\frac{1}{b_m}\frac{\partial U_m}{\partial P_i} \tag{12}$$

where U_m is the strain energy stored in the volume V_m due to the external loads P_i ($i=1,...,N_p$). If a finite element technique is used, the volume V_m is divided, in general, into a finite number of elements. In this case, U_m is the sum of all the element strain energies. Substituting Eq. (12) into Eq. (3) gives us

$$\frac{\partial g_j}{\partial b_m} = \frac{\partial G_j}{\partial b_m} - \frac{\partial G_j}{\partial u_i}\left(\frac{1}{b_m}\frac{\partial U_m}{\partial P_i}\right) \tag{13}$$

Note that Eq. (13) is of a form similar to Eq. (4).

By comparing Eq. (13) with Eq. (4), one can realize that the current method requires element strain energy terms in order to compute sensitivity coefficients, whereas the other methods require element stiffness matrices. Since the strain energy terms are commonly produced by finite element programs, the current method offers an advantage of easy implementation *without* modification to the source code.

Discussion and Conclusions

Before proceeding, it is useful to propose a procedure for implementing the present approach in conjunction with existing finite element programs (e.g., MSC/NASTRAN, etc.):

1) Choose a displacement component u_i of which one wishes to obtain derivatives with respect to b_m ($m=1,...,N_b$).

2) Apply two sets of loadings ($P_1...P_i...P_{N_p}$) and ($P_1,...,P_i+\Delta P_i,...,P_{N_p}$) to a structural model. It should be noted that other boundary conditions are identical except for the loading condition at point i. After completing the analysis, read strain energies for both loading cases [e.g., $U_e(P_i)$ and $U_e(P_i+\Delta P_i)$] from the computer output. Here $e=1,...,N_e$, where N_e denotes the total number of elements of a structural model.

3) Compute

$$\frac{\partial U_e}{\partial P_i} = \frac{U_e(P_i+\Delta P_i) - U_e(P_i)}{\Delta P_i} \tag{14}$$

for all elements, which depend on any of b_m.

4) Calculate

$$\frac{\partial}{\partial b_m}\left(\frac{\partial U_m}{\partial P_i}\right) = -\frac{1}{b_m}\sum_{e=1}^{N_m}\left(\frac{\partial U_e}{\partial P_i}\right) \tag{15}$$

where N_m ($<N_e$) is the number of elements that contain the mth design variable [see Eq. (13)].

5) Repeat steps 1-4 with a different displacement, u_{i+1}.

Step 2 suggests that two sets of loadinz cases are needed for each displacement u_i. Therefore, if there are N_u displacements in the constraint functions [Eq. (3)], one must solve $2N_u$ sets of loadings, which means, theoretically, $2N_u$ separate finite element analyses. The actual cost, however, is much less than $2N_u$ times the cost of a single finite element analysis, because the decomposed stiffness matrix K can usually be used for multiple loading cases (N_ℓ). Note that the cost of the present approach can be shown to be independent of the number of design variables N_b.

Thus far, an alternative method for computing design sensitivity coefficients based on Castigliano's theorem has been proposed. Although this theorem has been known for many years, its application to design sensitivity analysis is new. Since the cost of the present method is independent of the number of design variables, the efficiency appears to compare favorably with other methods. In addition, it offers advantages of both simple integration using existing finite element programs and a potential for extension to nonlinear problems.

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Methods of Reference Basis for Identification of Linear Dynamic Structures

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Introduction

THE initial approach to the identification of structures was to use test data only. For structures with a small number of degrees-of-freedom this approach can be satisfactory. However, for complex structures such as aircraft vehicles, test data are incomplete by necessity. In Refs. 1-6 it was proposed to add to the vibration test data independently obtained analytical mass, $A(n\times n)$, and stiffness, $K(n\times n)$, matrices. Now the available data are redundant and usually will not comply with the theoretical requirements of a physically possible structure. Hence, some of the data must be corrected. Some of the data can be taken as a reference basis and used to correct the remaining data.

The rigid body modes, $R(n\times r)$, are theoretically well defined and must be kept unchanged. The parts of the analytically obtained mass matrix A connected with the rigid body modes can be obtained in an independent manner and must also be kept unchanged.[5] It is widely accepted that the measured frequencies constitute the most accurate test data. Hence, the measured frequencies, represented by $\Omega^2(m\times m)$,

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are taken as the identified frequencies. Therefore, it follows that

$$AR = MR, \qquad R^T AR = R^T MR = I \tag{1}$$

where $M(n \times n)$ is the identified mass matrix.

The identified quantities must satisfy the basic vibration equation

$$MX\Omega^2 = YX \tag{2}$$

where $Y(n \times n)$ is the identified stiffness matrix and $X(n \times m)$ is the identified modal matrix. Also

$$X^t MX = I, \quad X^T YX = \Omega^2, \quad KR = YR = 0 \tag{3}$$

The first M.R.B. (method of reference basis), in which the mass is the reference basis,[1-4,6,7] and the second M.R.B., in which the measured modes are the reference basis,[5,6,8,9] already exist in the literature. In this Note a new method, in which the stiffness matrix is the reference basis, is proposed.

Third M.R.B.: Stiffness Matrix

General Considerations

Here the stiffness matrix is chosen to be the reference basis. This possibility was suggested in Ref. 10. However, to make the proposed method even more meaningful, the stiffness matrix itself will be corrected by using static tests. It must be emphasized that by using static tests the correction of the stiffness matrix is accomplished in a way which does not depend on the parameters of the structure. Partial solution for the correction of the stiffness matrix using static tests can be found in Ref. 11 where pseudoinverse of a singular matrix has been applied. In the proposed method, closed-form solutions have been found.

It is assumed that in addition to the vibration test results we have a static loads matrix $\tilde{F}(g \times f)$ and a static displacement matrix $\tilde{G}(g \times f)$, which for unconstrained structures is obtained by constraining the structure in a statically determinate way. Clearly, for unconstrained structures,

$$g = n - r \tag{4}$$

The loads applied on the structure can be measured in an independent way and will be considered as exact. Hence

$$F = \tilde{F} \tag{5}$$

Correction of the Displacement Matrix

The case of the measured displacement matrix $\tilde{G}(g \times f)$ is more complicated. In a complete structure one cannot measure all the displacements and some method must be found to supply the missing displacements. One simple method is to fill the gaps with displacements obtained from the analytical model by calculations. A second seemingly more sophisticated method follows:

$$\begin{bmatrix} \tilde{k}_1 & \tilde{k}_2 \\ \tilde{k}_2^t & \tilde{k}_4 \end{bmatrix} \begin{Bmatrix} \tilde{G}_i^l \\ \tilde{G}_i^2 \end{Bmatrix} \overset{?}{=} \begin{Bmatrix} F_i^l \\ F_i^2 \end{Bmatrix} \tag{6}$$

where $\tilde{k}(g \times g)$ is a given analytical stiffness matrix for the constrained structure, $\tilde{G}_i^l$ represents the measured displacements, $\tilde{G}_i^2$ represents the missing displacements, and F_i is a known load vector. Now, one cannot expect the measured displacements to satisfy the equality sign in Eq. (6). However, the equality sign can be imposed on the missing displacements to obtain

$$\tilde{G}_i^2 = \tilde{k}_4^{-1}(F_i^2 - \tilde{k}_2^t \tilde{G}_i^l) \tag{7}$$

Note that the second approach to find the missing displacements is similar to the process of reduction of the stiffness matrix given in Ref. 12. Following the general philosophy that measured data are better than analytical data, the second approach to find the missing displacements seems preferable. However, the opinion of the author is that, when possible, an approach in which only measured data are involved is even better. For example, this can be achieved by using a geometric interpolation method.[13]

We are now in possession of the load matrix $F(g \times f)$ and the connected displacement matrix $\tilde{G}(g \times f)$. Due to errors of different kinds, one cannot expect the two matrices to be compatible. In other words, the two matrices F and G cannot represent the loads and the displacements of a physically possible linear structure in their crude form. To do so they have to comply with the Maxwell-Betti reciprocal theorem. Here we have a case where corrected data are better than the raw data.[14] Expressed in matrix form the reciprocal theorem requires that the product $F^t G$ be symmetrical. This requirement can be fulfilled in the following way[6]:

$$F^t G = \tfrac{1}{2}(F^t \tilde{G} + \tilde{G}^t F) = G^t F \tag{8}$$

The natural norm of the displacement matrix seems to be the simple Euclidean norm and the distance between the corrected and measured displacements can be measured by

$$d = \tfrac{1}{2}\|G - \tilde{G}\| = \tfrac{1}{2}(g_{ij} - \tilde{g}_{ij})(g_{ij} - \tilde{g}_{ij}) \tag{9}$$

where the Einstein rule of summation is applied. The constraint, Eq. (8), can be incorporated in the norm, Eq. (9), by using Lagrange multipliers. Minimization of the so obtained Lagrange function with respect to G yields[6]

$$G = \tilde{G} + \tfrac{1}{2}F(F^t F)^{-1}(\tilde{G}^t F - F^t \tilde{G}) \tag{10}$$

The load matrix F must be of rank f.

Correction of the Stiffness Matrix Using Static Tests

The displacement matrix G and the load matrix F satisfy the reciprocal theorem and can be used now to correct the given analytical stiffness matrix $\tilde{k}(g \times g)$. It should be noted that $\tilde{k}(g \times g)$ belongs to the constrained structure. For the static case it seems that the Euclidean norm is the natural one and the distance between the correct stiffness matrix $k(g \times g)$ and the given analytical matrix $\tilde{k}(g \times g)$ can be measured by

$$e = \tfrac{1}{2}\|k - \tilde{k}\| = \tfrac{1}{2}(k_{ij} - \tilde{k}_{ij})(k_{ij} - \tilde{k}_{ij}) \tag{11}$$

The corrected matrix k has to satisfy the following constaints.

$$kG = F, \qquad k^t = k \tag{12}$$

By using Lagrange multipliers to incorporate the constraints one obtains a new Lagrange function. Minimization of this function with respect to k yields[6]

$$k = \tilde{k} - G(G^t G)^{-1}(G^t \tilde{k} - F^t)[I - \tfrac{1}{2}G(G^t G)^{-1}G^t]$$
$$- [I - \tfrac{1}{2}G(G^t G)^{-1}G^t](\tilde{k}G - F)(G^t G)^{-1}G^t \tag{13}$$

The corrected stiffness matrix of the unconstrained structure, $Y(n \times n)$, is given simply by

$$Y = \beta k \beta^t \tag{14}$$

where k is given by Eq. (13) and $\beta(n \times g)$ is the equilibrium matrix in which the vectors include a unit load in any one of the degrees-of-freedom of the constrained structure and the reactions of the statically determinate structure caused by the unit load.

Orthogonalization of the Measured Mode Shapes with Respect to the Stiffness Matrix

The case of measured flexible mode shapes obtained by vibration tests is similar to the case of displacements obtained by static tests. For a complex structure one cannot expect to have measurements in all the degrees-of-freedom. Here again the missing part of the mode shape can be supplied by several approaches. The first one is to fill the gaps by components obtained from the analytical model by calculations. The second approach is similar to the one given in Eq. (6) (see Refs. 9 and 15),

$$\left[\begin{bmatrix} Y_1 & Y_2 \\ Y_2^t & Y_4 \end{bmatrix} - \omega_i^2 \begin{bmatrix} A_1 & A_2 \\ A_2^t & A_4 \end{bmatrix}\right] \begin{Bmatrix} \tilde{T}_i^1 \\ \tilde{T}_i^2 \end{Bmatrix} \overset{?}{=} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15}$$

where ω_i is the measured frequency, $\tilde{T}_i^1$ the measured part of the mode shape, and $\tilde{T}_i^2$ the missing part of the mode shape. One cannot expect the the equality sign in Eq. (15) to be fully satisfied by the measured part of the mode shape. However, the equality sign can be imposed on the missing part of the mode shape to obtain[9]

$$\tilde{T}_i^2 = -(Y_4 - \omega_i^2 A_4)^{-1}(Y_2^t - \omega_i^2 A_2^t)\tilde{T}_i^1 \tag{16}$$

Again, the second method seems preferable because it involves more measured quantities. However, it seems even better, when possible, to use for example a gometric interpolation method[13] where only measured data are involved.

The corrected stiffness matrix Y [Eq. (14)] will be used as a reference basis to orthogonalize the measured flexible mode shapes and correct the mass matrix. Before that the modes must be slightly modified to meet a basic orthogonality condition. Here again we have a clear case where the corrected data are better than the raw data.[5,6,10,14] The flexible mode shape must be orthogonal to the theoretically well-defined rigid body mode shapes.

It is convenient to normalize the mode shapes in the following way:

$$T_i = \tilde{T}_i(\tilde{T}_i^t A \tilde{T}_i)^{-1/2} \tag{17}$$

By minimization of the norm

$$f = \tfrac{1}{2}\| N(\tilde{X} - T)\|, \qquad N = A^{1/2} \tag{18}$$

through satisfaction of the constraint

$$R^t A \tilde{X} = 0 \tag{19}$$

one obtains[5]

$$\tilde{X} = (I - RR^t A)T \tag{20}$$

Now, the stiffness matrix Y [Eq. (14)] is our reference basis and the measured flexible modes taken from Eq. (20) will be normalized as follows:

$$Q_i = \omega_i \tilde{X}_i(\tilde{X}_i^t Y \tilde{X}_i)^{-1/2} \tag{21}$$

The corrected flexible mode shapes $X(n \times m)$ must fulfill the following requirement [see Eq. (3)]:

$$X^t Y X = \Omega^2 \tag{22}$$

The distance between X and Q will be measured by[6]

$$h = \| L(X - Q)\|, \qquad L = Y^{1/2} \tag{23}$$

Minimization of h with respect to X with the constraint Eq. (22) yields

$$X = Q\Omega(\Omega Q^t Y Q \Omega)^{-1/2}\Omega \tag{24}$$

where only the positive definite solution of the negative square root must be taken into account.[1] Several methods for solution of this type of equations can be found in Refs. 1 and 4.

Equation (24) is valid also for the case of unconstrained structures due to the following considerations: Suppose that the structure is constrained by very soft springs and Q does not contain the almost rigid body modes. In this case Y is regular and the process of obtaining Eq. (24) is legal. By letting the spring coefficients approach zero the structure becomes unconstained, and due to reasons of continuity the validity of Eq. (24) is not changed.

Correction of the Mass Matrix Using the Stiffness Matrix as a Reference Basis

The corrected mass matrix must fulfill Eq. (2). In addition, it must be symmetric.

$$M^t = M \tag{25}$$

The corrected mass matrix should be as close as possible to the given analytical mass matrix. The closeness of the two matrices will be measured by the cost function[6]

$$s = \tfrac{1}{2}\| L^{-1}(M - A)L^{-1}\| \tag{26}$$

By using Lagrange multipliers to incorporate the constraints from Eqs. (2) and (25) one obtains a new Lagrange function. Minimization of this fucntion with respect to M yields[6]

$$M = A - AX\Omega^{-2}X^tY - YX\Omega^{-2}X^tA + YX\Omega^{-4}X^tY$$
$$+ YX\Omega^{-2}X^tAX\Omega^{-2}X^tY \tag{27}$$

The solution for the corrected mass matrix completes the third method of reference basis.

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Post-Buckling Behavior of a Thick Circular Plate

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Introduction

THE post-buckling behavior of circular plates has been the subject of a great deal of study for many years.[1,2] Wellford[3] and Rao[4] presented solutions for the post-buckling behavior of elastic circular plates by using a finite element method. The axisymmetric post-buckling behavior of plates has been recently tested experimentally.[5] All of the past investigators have been concerned mainly with in-plane compressive stresses and most of their studies were concerned with thin plate theory.

Using the average stress method, the authors have derived the nonlinear equations on the basis of von Kármán's assumption to study the large amplitude vibration[6] and post-buckling[7] problems for a thick rectangular plate in an arbitrary state of initial stress. In the present work, the previously derived nonlinear equations[8] of a transversely isotropic circular thick plate in a general state of nonuniform initial stress are used. The post-buckling problems of both simply supported and clamped axisymmetric circular plates subjected to uniform in-plane compression and a uniform bending stress acting along the edge are studied.

Governing Equations

Consider a circular plate of uniform thickness h and radius a in a state of arbitrary edge loading. The state of applied stress is

$$\sigma_{rr} = \sigma_n + 2z\sigma_m / h \tag{1}$$

σ_n and σ_m are taken to be constants. It is comprised of a compressive plus a bending stress. The only nonzero stress resultants are

$$N_r = h\sigma_n, \quad M_r = h^2\sigma_m/6, \quad M_r^* = h^3\sigma_n/12 \tag{2}$$

The coordinate system was chosen such that the middle plane of the plate coincides with the r-θ plane. The origin of the coordinate system begins at the center of the plate with the positive z axis upward.

For an axisymmetric circular plate, the θ dependence can be dropped and the displacement field is simplified by

$$\xi_r(r,z,t) = u(r,t) + z\psi_r(r,t), \quad \xi_\theta = 0$$
$$\xi_z(r,z,t) = w(r,t) \tag{3}$$

Lateral loads and body forces are taken to be zero. For the static problem, the equations of motion are as follows.[8]

$$\begin{aligned} &D(u_{,r}+\nu u/r)_{,r} + D(u_{,r}+\nu u/r)/r - D(u/r+\nu u_{,r})/r \\ &+ D(1-\nu)w_{,r}^2/2r + Dw_{,r}w_{,rr} + (N_r u_{,r} + M_r\psi_{r,r})_{,r} \\ &+ (N_r u_{,r} + M_r\psi_{r,r})/r = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} &(N_r w_{,r})_{,r} + N_r w_{,r}/r + \kappa^2 G^* h(w_{,r}+\psi_r)_{,r} + \kappa^2 G^* h(w_{,r}+\psi_r)/r \\ &+ D\epsilon_1(w_{,rr}+w_{,r}/r) + D\epsilon_{1,r}w_{,r} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &(M_r u + M_r^*\psi_{r,r})_{,r} + (M_r u_{,r} + M_r^*\psi_{r,r})/r \\ &+ D^*(\psi_{r,r}+\nu\psi_r/r)_{,r} + D^*(\psi_{r,r}+\nu\psi_r/r)/r \\ &- \kappa^2 G^* h(w_{,r}+\psi_r) - D^*(\psi_r/r+\nu\psi_{r,r})/r = 0 \end{aligned} \tag{6}$$

where

$$\epsilon_1 = u_{,r} + w_{,r}^2/2$$

The boundary conditions are for the simply supported immovable plate

$$\bar{M}_r + \Delta M_r = M_r u_{,r} + M_r^*\psi_{r,r} + D^*(\psi_{r,r}+\nu\psi_r/r) = 0$$
$$w = u = 0 \quad \text{at} \quad r = a$$
$$u = \psi_r = w_{,r} = 0 \quad \text{at} \quad r = 0 \tag{7}$$

and for the clamped immovable plate

$$w = u = \psi_r = 0 \quad \text{at} \quad r = a$$
$$u = \psi_r = w_{,r} = 0 \quad \text{at} \quad r = 0 \tag{8}$$

Displacements of the following form satisfy the geometric boundary conditions:

$$u(r,t) = u(A_1y + A_2y^3 + A_3y^5 + A_4y^7)$$
$$w(r,t) = w(1 + B_1y^2 + B_2y^4)$$
$$\psi_r(r,t) = \Psi(C_1y + C_2y^3) \tag{9}$$

where

$$y = r/a, \quad A_1 = (5-3\nu)/6, \quad A_2 = -(3-\nu)$$
$$A_3 = 2(5-\nu)/3, \quad A_4 = -(7-\nu)/6$$

For the simply supported plate

$$B_1 = -(6+2\nu)/(5+\nu), \qquad B_2 = (1+\nu)/(5+\nu)$$
$$C_1 = 2(6+2\nu)/(5+\nu), \qquad C_2 = -4(1+\nu)/(5+\nu)$$

and for the clamped plate

$$B_1 = -2, \quad B_2 = 1, \quad C_1 = 4, \quad C_2 = -4$$

Equations (10-12) are obtained by substituting the assumed displacement field of Eq. (9) into the equations of motion,

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